PHYSICAL REVIEW C **89**, 041302(R) (2014)

Linking partial and quasi dynamical symmetries in rotational nuclei

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(Received 10 November 2013; published 4 April 2014)

Background: Quasi dynamical symmetry (QDS) and partial dynamical symmetry (PDS) play an important role in the understanding of complex systems. Up to now these symmetry concepts have been considered to be unrelated.
Purpose: The goal of this paper is to establish a link between PDS and QDS and find an empirical manifestation.
Methods: Quantum number fluctuations and the intrinsic state formalism are used within the framework of the interacting boson model of nuclei.
Results: A previously unrecognized region of the parameter space of the interacting boson model that has both O(6) PDS (purity) and SU(3) QDS (coherence) in the ground band is established. Many rare-earth nuclei approximately satisfying both symmetry requirements are identified.
Conclusions: PDS is more abundant than previously recognized and can lead to a QDS of an incompatible symmetry.

DOI: 10.1103/PhysRevC.89.041302 PACS number(s): 21.60.Fw, 21.10.Re, 21.60.Ev, 27.70.+q

Understanding the structure and dynamics of complex many-body systems can often be obtained from the observation and analysis of symmetries. Symmetry considerations are particularly significant for addressing a key question in such systems, namely, how do simple features emerge within a complicated environment. A notable example is the collective behavior of nuclei which stems from the complex interactions among the constituent nucleons. Despite the complex nature of the low-energy effective forces at work and the large number of participating particles, collective nuclei give rise to strikingly regular excitation spectra, signaling the presence of underlying symmetries [1]. The theme of "simplicity out of complexity" and the understanding of simple emergent behavior are major challenges facing the study of almost any many-body system, from atomic nuclei to nanoscale and macroscopic systems [2].

Although, usually, a many-body Hamiltonian does not conform to a dynamical symmetry (DS) limit [3], the possibility exists that certain symmetries are obeyed by only a subset of its eigenstates. This situation, referred to as partial dynamical symmetry (PDS) [4], was shown to be relevant to specific nuclei [4–12] and molecules [13]. In parallel, the notion of quasi dynamical symmetry (QDS) was introduced and discussed in the context of nuclear models [14–21]. While QDS can be defined mathematically in terms of embedded representations [22,23], its physical meaning is that several observables associated with particular eigenstates may be consistent with a certain symmetry which in fact is broken in the Hamiltonian. This typically occurs for a Hamiltonian transitional between two DS limits which retains, for a certain range of its parameters, the characteristics of one of those limits. This "apparent" symmetry is due to a coherent mixing of representations in selected states, imprinting an adiabatic motion and increased regularity [19–21].

PDS and QDS are applicable to any many-body problem (bosonic and fermionic) endowed with an algebraic structure. They play a role in diverse phenomena including nuclear and molecular spectroscopy [4–16], quantum phase transitions [17–19,24], and mixed regular and chaotic dynamics [20,21,25]. In this Rapid Communication, a hitherto unnoticed link is established between these two different symmetry concepts and it is shown that coherent mixing of one symmetry (QDS) can result in the partial conservation of a different, incompatible symmetry (PDS). An empirical manifestation of such a linkage is presented.

Algebraic models provide a convenient framework for exploring the role of symmetries [26]. One such framework is the interacting boson model (IBM) [27], which has been widely used to describe quadrupole collective states in nuclei in terms of N monopole ($s^\dagger$) and quadrupole ($d^\dagger$) bosons, representing valence nucleon pairs. The model has U(6) as a spectrum generating algebra and exhibits three DS limits, associated with chains of nested subalgebras, starting with U(5), O(6), and SU(3), respectively. These solvable limits correspond to known benchmarks of the geometric description of nuclei [28], involving vibrational [U(5)], γ-soft [O(6)], and rotational [SU(3)] types of dynamics. In what follows we employ the IBM as a test ground for connecting the PDS and QDS notions. The particular example considered, namely, SU(3) QDS as an emanation of O(6) PDS, is shown to have approximate validity in many deformed rare-earth nuclei.

One particularly successful approach within the IBM is the extended consistent-Q formalism (ECQF) [29,30], which is frequently used for the interpretation and classification of nuclear data. It uses the same quadrupole operator, $\hat{Q}^\chi = d^\dagger s + s^\dagger \tilde{d} + \chi\,(d^\dagger \tilde{d})^{(2)}$, in the $E2$ transition operator and in the Hamiltonian, the latter being written as

$$\hat{H}_{\rm ECQF} = \omega\left[(1-\xi)\,\hat{n}_d - \frac{\xi}{4N}\,\hat{Q}^\chi\cdot\hat{Q}^\chi\right], \qquad (1)$$

where $\hat{n}_d$ is the d-boson number operator, $\hat{Q}^\chi \cdot \hat{Q}^\chi$ is the quadrupole interaction, and the dot implies a scalar product. The parameters ω, ξ, and χ are fitted to empirical data or calculated microscopically if possible; ξ and χ are the sole structural parameters of the model since ω is a scaling factor. The parameter ranges $0 \leqslant \xi \leqslant 1$ and $-\frac{\sqrt{7}}{2} \leqslant \chi \leqslant 0$ interpolate between the U(5), O(6), and SU(3) DS limits, which are reached for $(\xi,\chi) = (0,\chi)$, $(1,0)$, and $(1,-\frac{\sqrt{7}}{2})$, respectively. It is customary to represent the parameter space by a symmetry triangle [31], whose vertices correspond to these limits. The ECQF has been used extensively for the description of nuclear properties (see, e.g., Ref. [32]) and it was found that rotational nuclei are best described by ECQF parameters in the interior of the triangle, away from the naively expected SU(3) DS limit. The SU(3) mixing was found to be strong and coherent, i.e., the same for all rotational states in a band, exemplifying a SU(3) QDS [19–21]. In what follows we examine the O(6) symmetry properties of ground-band states in such nuclei, in the rare-earth region, using the ECQF of the IBM.

The O(6) DS basis states are specified by quantum numbers N, σ, τ, and L, related to the algebras in the chain $U(6) \supset O(6) \supset O(5) \supset O(3)$ [33]. Given an eigenstate Ψ of the ECQF Hamiltonian (1), its expansion in the O(6) basis reads

$$|\Psi(\xi,\chi)\rangle = \sum_i \alpha_i(\xi,\chi)\,|N,\sigma_i,\tau_i,L\rangle, \tag{2}$$

where the sum is over all basis states and, for simplicity, the dependence of Ψ and α_i on the boson number N and the angular momentum L is suppressed. The degree of O(6) symmetry of the state Ψ is inferred from the fluctuations in σ which can be calculated as

$$\Delta\sigma_\Psi = \sqrt{\sum_i \alpha_i^2\,\sigma_i^2 - \left(\sum_i \alpha_i^2\,\sigma_i\right)^2}. \tag{3}$$

If Ψ carries an exact O(6) quantum number, σ fluctuations are zero, $\Delta\sigma_\Psi = 0$. If Ψ contains basis states with different O(6) quantum numbers, then $\Delta\sigma_\Psi > 0$, indicating that the O(6) symmetry is broken. Note that $\Delta\sigma_\Psi$ also vanishes for a state with a mixture of components with the same σ but different O(5) quantum numbers τ, corresponding to a Ψ with good O(6) but mixed O(5) character. This method of quantifying the O(6) purity of states has already been applied to ^{124}Xe [34]. Also, $\Delta\sigma_\Psi$ has the same physical content as wave-function entropy which, upon averaging over all eigenstates, discloses the global DS content of a given Hamiltonian [35]. We examine here the fluctuations $\Delta\sigma_\Psi$ for the entire parameter space of the ECQF Hamiltonian (1) for values of N up to 60, using the ARBMODEL code [36].

Results of this calculation for the ground state, $\Psi = 0^+_{\rm g.s.}$, with $N = 14$ and parameters $\xi \in [0,1]$, $\chi \in [-\frac{\sqrt{7}}{2},0]$, are shown in Fig. 1. At the O(6) DS limit ($\xi = 1$, $\chi = 0$) $\Delta\sigma_{\rm g.s.}$ vanishes per construction whereas it is greater than zero for all other parameter pairs. Towards the U(5) DS limit ($\xi = 0$), the fluctuations reach a saturation value of $\Delta\sigma_{\rm g.s.} \approx 2.47$. At the SU(3) DS limit ($\xi = 1$, $\chi = -\frac{\sqrt{7}}{2}$) the fluctuations are $\Delta\sigma_{\rm g.s.} \approx 1.25$. In both cases the O(6) symmetry is completely dissolved as measured by $\sigma_{\rm crit} = 0.849$ [34]. Surprisingly, there is a previously unrecognized valley of almost vanishing $\Delta\sigma_{\rm g.s.}$ values, two orders of magnitude lower than at saturation. This region represents a parameter range of the IBM, outside the O(6) DS limit, where the ground-state wave function exhibits an exceptionally high degree of purity with respect to the O(6) quantum number σ.

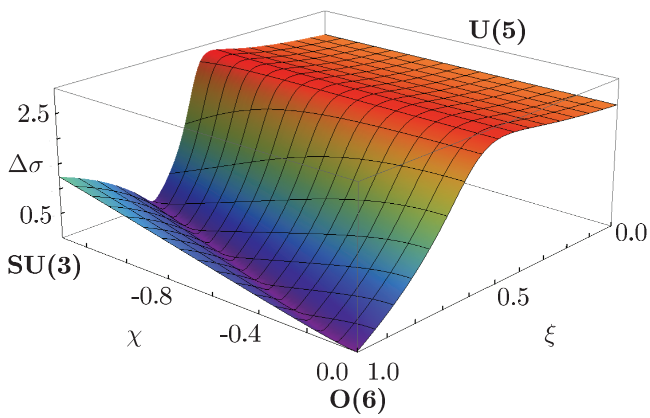

FIG. 1. (Color online) Ground-state fluctuations $\Delta\sigma_{\rm g.s.}$ (3) for the ECQF Hamiltonian (1) with $N = 14$ bosons. The fluctuations vanish at the O(6) DS limit, saturate towards the U(5) DS limit, and are of the order 10^{-2} in the valley.

The ground-state wave functions in the valley of low $\Delta\sigma_{\rm g.s.}$ can be analyzed with the help of the O(6) decomposition (2). At the O(6) DS limit, only one O(6) basis state with $\sigma = N$ and $\tau = 0$ contributes, while outside this limit the wave function consists of multiple O(6) basis states. Investigation of the wave function for parameter combinations inside the valley reveals an overwhelming dominance of the O(6) basis states with $\sigma = N$. This is seen in Fig. 2 for the ground-state wave function of the ECQF Hamiltonian (1) at $\xi = 0.84$ and $\chi = -0.53$ with $N = 14$, parameter values that apply to the nucleus ^{160}Gd discussed below. The $\sigma = N$ states comprise more than 99%

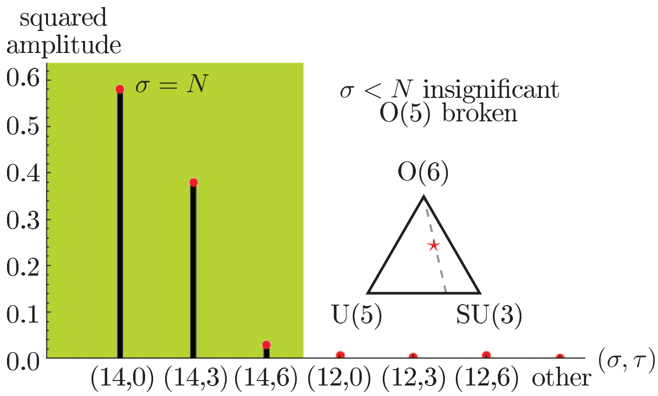

FIG. 2. (Color online) Squared amplitudes α_i^2 in the expansion (2) of the $0^+_{\rm g.s.}$ ground state of the ECQF Hamiltonian (1) for $\xi = 0.84$ and $\chi = -0.53$ (indicated by the red star in the symmetry triangle and appropriate for ^{160}Gd).

of the ground-state wave function at the bottom of the valley and their dominance causes $\Delta\sigma_{\text{g.s.}}$ to be small. Furthermore, it is evident that at the same time the O(5) symmetry is broken, as basis states with different quantum number τ contribute significantly to the wave function. Consequently, the valley can be identified as an entire region in the symmetry triangle with an approximate PDS of type III [4], which means that some of the eigenstates exhibit some of the symmetries. Outside this valley the ground state is a mixture of several σ values, and $\Delta\sigma_{\text{g.s.}}$ increases. In the SU(3) DS limit the $\sigma = N$ components constitute 67% of the wave function and in the U(5) DS limit and throughout the plateau of saturated $\Delta\sigma_{\text{g.s.}}$ this contribution drops below 1%. This region of approximate ground-state O(6) symmetry is similar to the previously established "arc of regularity" [37] which is a region of reduced mixing inside the IBM parameter space attributed to an approximate SU(3) symmetry [38].

An argument for the existence of the valley of ground-state O(6) symmetry can be given in terms of the following Hamiltonian [7]:

$$\hat{H}_{\text{M}} = -\hat{C}_{\text{O(6)}} + \hat{N}(\hat{N}+4) + 2\alpha\hat{C}_{\text{O(5)}} - \alpha\hat{C}_{\text{O(3)}} + 2\alpha\hat{n}_d(\hat{N}-2) + \sqrt{14}\alpha(d^\dagger s + s^\dagger\tilde{d})\cdot(d^\dagger\tilde{d})^{(2)}, \quad (4)$$

where $\hat{C}_G$ denotes the quadratic Casimir operator of the group G [27], $\hat{N}$ is the total boson number operator, and α is a parameter. The Hamiltonian (4) generates a PDS of type III [4]. For $\alpha = 0$, $\hat{H}_{\text{M}}$ has exact O(6) symmetry whereas for $\alpha > 0$ the last two terms introduce O(6)-symmetry breaking. However, the yrast states of this Hamiltonian, projected from the IBM intrinsic state with shape variables [28], $\beta = 1$ and $\gamma = 0$, keep exact O(6) symmetry ($\sigma = N$) but break the O(5) symmetry (mixed τ) for all values of $\alpha > 0$ [7]. Interestingly, although $\hat{H}_{\text{M}}$ differs from $\hat{H}_{\text{ECQF}}$, the overlap between their $0^+_{\text{g.s.}}$ ground states maximizes (more than 99%) in extended regions of (ξ,χ) inside the valley of low $\Delta\sigma_{\text{g.s.}}$. This suggests that the $(\beta = 1, \gamma = 0)$ intrinsic state provides a good approximation, in a variational sense, to the ground band of $\hat{H}_{\text{ECQF}}$ along the valley. The equilibrium deformations for a given IBM Hamiltonian are found by minimizing an energy surface, $E(\beta,\gamma)$, obtained by its expectation value in an intrinsic state which is a condensate of N bosons, $b_c^\dagger \propto \beta\cos\gamma d_0^\dagger + \beta\sin\gamma(d_2^\dagger + d_{-2}^\dagger)/\sqrt{2} + s^\dagger$, that depends parametrically on (β,γ) [39,40]. Apart from a constant, $E(\beta,\gamma) \propto (1+\beta^2)^{-2}\beta^2[a - b\beta\cos 3\gamma + c\beta^2]$, where a, b, and c are coefficients depending on the Hamiltonian. The two extremum equations, $\partial E/\partial\beta = \partial E/\partial\gamma = 0$, have $\beta = 1$ and $\gamma = 0$ as a solution, provided $b = 2c$. For large N, the coefficients of $\hat{H}_{\text{ECQF}}$ are $b = -\omega\xi\sqrt{\frac{2}{7}}\chi/N$ and $c = \omega[1 - \xi - \xi\chi^2/14]/N$. Thus, in the valley of low $\Delta\sigma_{\text{g.s.}}$ the desired condition, $b = 2c$, fixes ξ to be

$$\xi = \frac{1}{1 - \sqrt{\frac{1}{14}}\chi + \frac{1}{14}\chi^2}. \quad (5)$$

As seen in Fig. 3, this relation predicts the location of the region of approximate ground-state O(6) symmetry for large N very precisely. For small N its precision decreases somewhat due to finite-N effects, causing a more pronounced curvature of the region close to the O(6) limit.

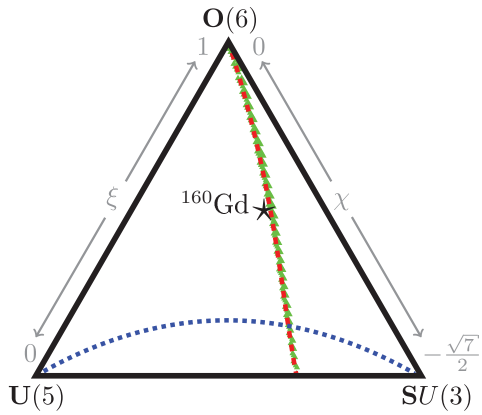

FIG. 3. (Color online) The ECQF symmetry triangle with the position of the nucleus ^{160}Gd indicated by a star. The green area shows the region of low $\Delta\sigma_{\text{g.s.}}$, calculated from Eq. (3) for $N = 60$. The red dashed line shows the same region of approximate ground-state O(6) symmetry, as predicted by Eq. (5) for large N. The blue dotted line shows the "arc of regularity" [37].

Detailed ECQF fits for energies and electromagnetic transitions of rare-earth nuclei, performed by McCutchan *et al.* [32], allow one to relate the structure of collective nuclei to the parameter space of the ECQF Hamiltonian (1). Examining the extracted (ξ,χ) parameters, one finds that several rotational nuclei in this region, such as ^{160}Gd, commonly interpreted as SU(3)-like nuclei, are actually located in the valley of small σ fluctuations. They can be identified as candidate nuclei with approximate ground-state O(6) symmetry. The experimental spectrum of ^{160}Gd, along with its ECQF description with $\xi = 0.84$ and $\chi = -0.53$ taken from Ref. [32], is shown in Fig. 4(a). Figures 4(b) and 4(c) show the decomposition into O(6) and SU(3) basis states, respectively, for yrast states with $L = 0,2,4$. It is evident that the SU(3) symmetry is broken, as significant contributions of basis states with different SU(3) quantum numbers (λ,μ) occur. It is also clear from Fig. 4(c)

TABLE I. Calculated σ fluctuations $\Delta\sigma_L$, Eq. (3), for rare-earth nuclei in the vicinity of the identified region of approximate ground-state O(6) symmetry. Also shown are the fraction $f^{(L)}_{\sigma=\text{N}}$ of O(6) basis states with $\sigma = N$ contained in the $L = 0,2,4$ states, members of the ground band. The structure parameters ξ and χ are taken from [32].

Nucleus	N	ξ	χ	$\Delta\sigma_0$	$f^{(0)}_{\sigma=N}$	$\Delta\sigma_2$	$f^{(2)}_{\sigma=N}$	$\Delta\sigma_4$	$f^{(4)}_{\sigma=N}$
^{156}Gd	12	0.72	−0.86	0.46	95.3%	0.43	95.8%	0.38	96.6%
^{158}Gd	13	0.75	−0.80	0.35	97.2%	0.33	97.5%	0.30	97.9%
^{160}Gd	14	0.84	−0.53	0.19	99.1%	0.19	99.2%	0.17	99.3%
^{162}Gd	15	0.98	−0.53	0.41	96.0%	0.40	96.0%	0.40	96.1%
^{160}Dy	14	0.81	−0.49	0.44	96.2%	0.39	96.4%	0.36	96.8%
^{162}Dy	15	0.92	−0.31	0.07	99.9%	0.07	99.9%	0.06	99.9%
^{164}Dy	16	0.98	−0.26	0.13	99.6%	0.13	99.6%	0.13	99.6%
^{164}Er	14	0.84	−0.37	0.39	96.5%	0.37	96.7%	0.35	97.1%
^{166}Er	15	0.91	−0.31	0.12	99.7%	0.11	99.7%	0.10	99.7%

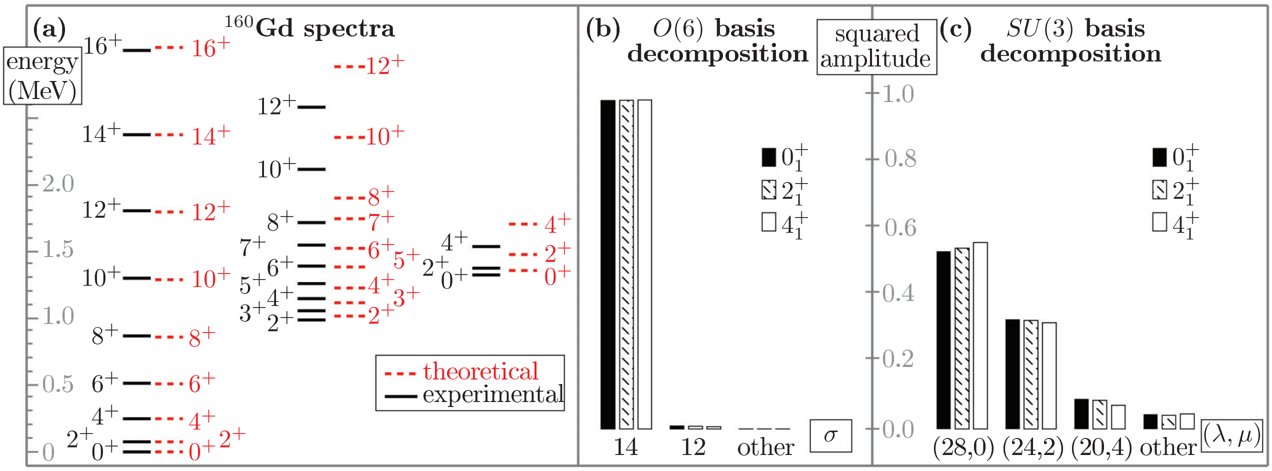

FIG. 4. (Color online) (a) The experimental spectrum of ^{160}Gd compared with the IBM calculation using the ECQF Hamiltonian (1) with parameters $\xi = 0.84$ and $\chi = -0.53$ taken from Ref. [32]. (b) The O(6) decomposition in σ components of yrast states with $L = 0,2,4$. (c) The SU(3) decomposition in (λ,μ) components of the same yrast states.

that this mixing occurs in a coherent manner with similar patterns for the different members of the ground-state band. This is the hallmark of a QDS [18] and it results from the existence of a single intrinsic wave function for the members of this band. On the other hand, as seen in Fig. 4(b), the yrast states with $L = 0,2,4$ are almost entirely composed of O(6) basis states with $\sigma = N = 14$ which implies small fluctuations $\Delta\sigma_\Psi$ and the preservation of O(6) symmetry in the ground-state band.

Other rare-earth nuclei with ground-state bands with approximate O(6) symmetry can be identified by the same arguments. Their structure parameters ξ and χ can be taken from Ref. [32], from where the fluctuations $\Delta\sigma_\Psi$ and the fractions $f_{\sigma=N}$ of squared $\sigma = N$ amplitude can be calculated. Nuclei with $\Delta\sigma_{\mathrm{g.s.}} < 0.5$ and $f_{\sigma=N} > 95\%$ are listed in Table I. These quantities are also calculated for yrast states with $L > 0$ and exhibit similar values in each nucleus. It is evident that the IBM predicts a high degree of O(6) purity in the ground-state band, for a large set of rotational rare-earth nuclei.

These results show that the approximate O(6) PDS does hold not only for the ground state but also for the members of the band built on top of it. Since the entire band corresponds to a single intrinsic state, the SU(3) wave-function decomposition is similar for the different members of the band and therefore the notion of SU(3) QDS applies. In addition, provided the indicated intrinsic state has $\beta \approx 1$ and $\gamma = 0$, the notion of O(6) PDS applies. Thus a link is established between SU(3) QDS and O(6) PDS.

To summarize, the method of quantum-number fluctuations reveals the existence of a region of almost exact ground-state-band O(6) symmetry outside the O(6) DS limit of the IBM. The existence of a valley of small σ fluctuations can be understood in terms of an approximate O(6) PDS of type III. The same wave functions display coherent (L-independent) mixing of SU(3) representations and hence comply with the conditions of an SU(3) QDS. Coherent mixing of one symmetry may therefore result in the purity of a quantum number associated with partial conservation of a different, incompatible symmetry. Previously established ECQF systematics show that many rare-earth nuclei do exhibit these approximate partial O(6) and quasi-SU(3)-dynamical symmetries. We conclude that partial dynamical symmetries are more abundant than previously recognized, may lead to coherent mixing and quasi dynamical symmetries, and hence play a role in understanding the regular behavior of complex nuclei. This example serves to illustrate a fundamental linkage between two distinct types of intermediate symmetries, PDS and QDS, with potential implications to algebraic modeling of diverse dynamical systems.

This work has been supported by the DFG through Grant No. SFB634 and, in part, by the Israel Science Foundation (A.L.).

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