Quantum Chaos, Irreversible Classical Dynamics, and Random Matrix Theory

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The Bohigas-Giannoni-Schmit conjecture stating that the statistical spectral properties of systems which are chaotic in their classical limit coincide with random matrix theory (RMT) is proved. A new semiclassical field theory for individual chaotic systems is constructed in the framework of a nonlinear σ model. The low lying modes are shown to be associated with the Perron-Frobenius (PF) spectrum of the underlying irreversible classical dynamics. It is shown that the existence of a gap in the PF spectrum results in RMT behavior. Moreover, our formalism offers a way of calculating system specific corrections beyond RMT. [S0031-9007(96)00191-3]

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Random matrix theory (RMT) [1] emerged from the need to characterize complex quantum systems in which knowledge of the Hamiltonian is minimal, e.g., complex nuclei. The basic hypothesis is that the Hamiltonian may be treated as one drawn from an ensemble of random matrices with appropriate symmetries. It has been proposed by invoking the complexity of systems which have many degrees of freedom with unknown interaction coupling among them.

The study of the statistical quantum properties of systems with a *small number* of degrees of freedom and their relation to RMT has developed along two lines. The first was by considering an ensemble of random systems such as disordered metallic grains [2]. Randomness in this case is introduced on the level of the Hamiltonian itself, often as a consequence of an unknown impurity configuration. In the second approach, RMT was used in order to understand the level statistics of nonstochastic systems which are chaotic in their classical limit such as the Sinai or the stadium billiards [3]. Here "randomness" is generated by the underlying deterministic classical dynamics itself. Nevertheless, it has been conjectured [3] that "spectrum fluctuations of quantal time-reversal-invariant systems whose classical analogs are strongly chaotic have the Gaussian orthogonal ensembles pattern."

Despite being supported by extensive numerical studies, the origin of the success of RMT as well as its domain of validity are still not completely resolved. In this Letter we show that, in the semiclassical limit, this conjecture is indeed valid for systems with exponential decay of classical correlation functions in time. Moreover, the formalism which we introduce offers a way of calculating system specific corrections beyond RMT.

So far the main attempts to establish the relationship between nonstochastic chaotic systems and RMT have been based on *periodic orbit theory* [4]. Gutzwiller's trace formula expresses the density of states (DOS) as sum over the classical periodic orbits of the system. However, the number of relevant periodic orbits is exponentially large and clearly contains information that is redundant from the quantum mechanical point of view. This detailed information conceals the way of drawing a connection between the quantum behavior of chaotic systems and RMT. Indeed, the success of the periodic orbit theory approach in reproducing RMT results [5,6] appears to be limited.

Here we develop a new semiclassical approach in which the basic ingredients are global modes of the time evolution of the underlying classical system rather than periodic orbits. It is possible to construct a field theory in which the effective action is associated with the classical flow in phase space. We argue that the statistical quantum properties of the system are intimately related to the *irreversible* classical dynamics or, more precisely, to the Perron-Frobenius (PF) modes in which a disturbance in the classical probability density of a chaotic system relaxes into the ergodic distribution. These modes decay at different rates. This enables a description of the system at levels of increasing complexity by incorporating higher and higher modes. The "zero mode" manifests the conservation of classical probability and corresponds to a uniform distribution over the energy shell. Taking into account only this mode one obtains RMT. Deviations from the universal RMT behavior emerge from the consideration of the higher PF modes.

Our approach is analogous to that of disordered systems where the diffusion modes account for the classical relaxation. However, in the field theoretic description of disordered systems [7] averaging is performed over an ensemble. By contrast, in order to characterize *individual* systems, only energy averaging will be employed here.

To establish the Bohigas-Giannoni-Schmit (BGS) conjecture we first show that quantum statistical correlators are described by a functional nonlinear σ model. Its low lying modes are identified with the PF eigenmodes of the underlying classical dynamics. We then argue that, provided classical correlation functions decay exponentially in time, there is an energy domain where the zero mode contribution governs the behavior. This follows from the fact that in such systems the PF spectrum has a gap. Finally we establish the relation to RMT by identifying it as the zero mode of the constructed field theory.

Let $\hat{H}$ define a quantum Hamiltonian having no discrete symmetries and whose classical counterpart, $H(\mathbf{x})$, is chaotic. Here we adopt the notation $\mathbf{x} = (\mathbf{q}, \mathbf{p})$ to denote a vector defined in a $2 \times d$ dimensional phase space. We restrict attention to a system in which all classical trajectories are unstable, and there are no islands of regular motion in phase space. At sufficiently high energy E_0, the mean spacing Δ between adjacent energy levels can be assumed constant and expressed through the Weyl formula,

$$\frac{1}{\Delta} = \frac{1}{(2\pi\hbar)^d} \int d^{2d}x\, \delta[E - H(\mathbf{x})]. \tag{1}$$

Henceforth energy and time will be measured in units of Δ and the Heisenberg time $\hbar/\Delta$, respectively.

The energy averaging is taken over an energy band containing a large number of levels N such that $1 \ll N \ll \epsilon_0 = E_0/\Delta$. For simplicity we choose to work with Gaussian averages ($\epsilon = E/\Delta$):

$$\langle \cdots \rangle_{\epsilon_0} = \int \frac{d\epsilon}{\sqrt{2\pi}\, N} \exp\left[-\frac{(\epsilon - \epsilon_0)^2}{2N^2} \right] (\cdots). \tag{2}$$

The basic quantity calculated within the field theoretic approach is the generating function $\langle \mathcal{Z}(J) \rangle_{\epsilon_0}$. Any n-point correlation function can be obtained by taking its derivatives with respect to the various components of the source J. To keep the discussion simple, we will restrict attention to systems belonging to the unitary ensemble (i.e., those with broken T invariance) and focus on two-point correlation functions such as the correlator of DOS, $R(s) = \langle \rho(\epsilon + s)\rho(\epsilon) \rangle_{\epsilon_0} - 1$, where $\rho(\epsilon) = \mathrm{Tr}\,\delta(\epsilon - \hat{H})$. As long as s is sufficiently small compared to the bandwidth N, the final results are independent of the particular form of energy averaging.

One can express $\langle \mathcal{Z}(J) \rangle_{\epsilon_0}$ as a functional integral in the usual way [7]. Introducing the four component supervector field $\Psi^T(\mathbf{q}) = (\psi^R, \chi^R, \psi^A \chi^A)$, where ψ (χ) denote commuting (anticommuting) components, and the superscript A (R) denotes the advanced (retarded) components [7], we can define

$$\mathcal{Z}(J) = \int \mathcal{D}\Psi\, e^{-S[\Psi,\epsilon]}, \tag{3}$$

where the action is given by

$$S[\Psi, \epsilon] = i \int d^d q\, \bar{\Psi}(\mathbf{q}) \left[\epsilon - \frac{s^+}{2}\Lambda - \hat{H} - Jk\Lambda \right] \times \Psi(\mathbf{q}). \tag{4}$$

Here $s^+ = s + i0$, while $\Lambda = \mathrm{diag}\,(1, 1, -1, -1)$ and $k = \mathrm{diag}\,(1, -1, 1, -1)$ break the symmetry between retarded or advanced field components and supersymmetry, respectively, and $\bar{\Psi} = \Psi^\dagger \Lambda$. The use of fermionic as well as bosonic components ensures the normalization $\mathcal{Z}(0) = 1$. To evaluate the two-point density correlator one can choose J to be constant, then

$$R(s) = -\frac{1}{16\pi^2} \Re\, \frac{\partial^2 \langle \mathcal{Z}(J) \rangle_{\epsilon_0}}{\partial J^2} \bigg|_{J=0}. \tag{5}$$

Energy averaging (2) of $\mathcal{Z}(J)$ generates a quartic interaction of the Ψ fields: $S(\Psi, \epsilon) \to S(\Psi, \epsilon_0) + S_i$, where

$$S_i = \frac{N^2}{2} \left(\int d^d q\, \bar{\Psi}(\mathbf{q}) \Psi(\mathbf{q}) \right)^2. \tag{6}$$

In contrast to impurity averaging [7], energy averaging induces a *nonlocal* interaction of the Ψ fields. This interaction term can be decoupled by means of the Hubbard-Stratonovich transformation, with the introduction of 4×4 supermatrix fields $Q(\mathbf{q}, \mathbf{q}')$ which depend on two coordinates,

$$\exp(-S_i) = \int \mathcal{D}Q \exp\left[-\mathrm{STr}_{\mathbf{q}} \left(\frac{Q^2}{2} - iNQ\Psi\bar{\Psi} \right) \right]. \tag{7}$$

Here $\mathrm{STr}_{\mathbf{q}}$ denotes the trace operation for supermatrices, while the subscript $\mathbf{q}$ implies a further extension of the trace to include integration over all spatial variables, e.g., $\mathrm{STr}_{\mathbf{q}} Q^2 = \int d^d q d^d q'\, \mathrm{STr}\, Q(\mathbf{q}, \mathbf{q}') Q(\mathbf{q}', \mathbf{q})$. Integrating over Ψ we obtain

$$\langle \mathcal{Z}(J) \rangle_{\epsilon_0} = \int \mathcal{D}Q \exp\left(-\frac{1}{2} \mathrm{STr}_{\mathbf{q}} Q^2 + \mathrm{STr}_{\mathbf{q}} \ln \mathcal{G} \right), \tag{8}$$

$$\mathcal{G}^{-1}(Q) = \epsilon_0 - \frac{s^+}{2}\Lambda - \hat{H} - Jk\Lambda - NQ. \tag{9}$$

Further progress is possible only within a saddle-point approximation which relies on an expansion in $1/N$. Varying the total action with respect to Q one obtains the saddle-point equation,

$$Q_0 \mathcal{G}^{-1}(Q_0) = N, \tag{10}$$

where Q_0 and $\mathcal{G}(Q_0)$ are operators.

To understand the structure of the saddle-point manipfold it is useful to employ the Wigner representation of operators. Given an operator $\hat{\mathcal{O}}$ as a set of matrix elements $\mathcal{O}(\mathbf{q}_1, \mathbf{q}_2)$ between two position states at $\mathbf{q}_1$ and $\mathbf{q}_2$, its Wigner representation is a function of the phase space variables $\mathbf{x}$ defined by $\mathcal{O}(\mathbf{x}) = \int d^d q' \exp(i\mathbf{p}\mathbf{q}'/\hbar) \mathcal{O}(\mathbf{q} + \mathbf{q}'/2, \mathbf{q} - \mathbf{q}'/2)$. We will use the result that, in the semiclassical limit, the Wigner transform of a product of operators is equal to the product of the Wigner transformed operators, $(\mathcal{O}_1\mathcal{O}_2)(\mathbf{x}) \to \mathcal{O}_1(\mathbf{x})\mathcal{O}_2(\mathbf{x})$, where $\mathcal{O}_{1,2}(\mathbf{x})$ are smooth slowly varying functions on the quantum scale.

Treating s and J as small compared to the bandwidth N, and introducing the phase space variables, $\mathbf{x}_\parallel$ on the energy shell and $x_\perp \equiv H(\mathbf{x})$ perpendicular to the energy shell, the solution of Eq. (10) can be expressed in the Wigner representation as

$$Q_0(\mathbf{x}) = \frac{\epsilon_0 - H}{2N} + i\left[1 - \left(\frac{\epsilon_0 - H}{2N}\right)^2\right]^{1/2}\Lambda\,. \quad (11)$$

However, this solution is not unique. In fact, the saddle-point solutions form a degenerate manifold in superspace associated with the group of pseudounitary rotations. Both the integration measure and the action in Eq. (8) are invariant under the group of transformations $Q(\mathbf{x}) \to U^{-1}(x_\perp)Q(\mathbf{x})U(x_\perp)$, where $U(x_\perp)$ belongs to the pseudounitary supergroup $U(1,1/2)$ [8]. Thus, any matrix of the form $Q(x_\perp) = U^{-1}(x_\perp)Q_0(x_\perp)U(x_\perp)$ is a solution of the saddle-point equation (8).

When integrating over the fluctuations δQ, near the saddle-point manifold, one has to take into account the anisotropy of the dependence of the action on the fluctuations δQ. Those fluctuations on which the action depends strongly (massive modes) can be integrated out within a conventional saddle-point approximation. The remaining fluctuations describe the Goldstone modes of the system, and their integration must be performed exactly. These degrees of freedom can be parametrized by $Q(\mathbf{x}) = T^{-1}(\mathbf{x})Q_0(x_\perp)T(\mathbf{x})$. However, integration over the massive modes [9] shows that, in the limit $N \gg 1$, the only nonvanishing contribution comes from matrices T which are independent of the energy $x_\perp$. The Goldstone modes can therefore be parametrized by

$$Q(\mathbf{x}) = T^{-1}(\mathbf{x}_\parallel)Q_0(x_\perp)T(\mathbf{x}_\parallel)\,, \quad (12)$$

where $T(x_\parallel)$ belong to the coset space $U(1,1/2)/U(1/1) \times U(1/1)$ [8].

The derivation of the effective field theory can be obtained by (i) substituting Eq. (12) into Eq. (8), approximating commutators by Poisson brackets $[\mathcal{O}_1, \mathcal{O}_2](\mathbf{x}) \to i\hbar\{\mathcal{O}_1(\mathbf{x}), \mathcal{O}_2(\mathbf{x})\}$, and replacing the trace by the phase space integral, $\mathrm{Tr}_{\mathbf{q}}(\mathcal{O}) = h^{-d}\int d\mathbf{x}\,\mathcal{O}(\mathbf{x})$ (this is the entry point of the semiclassical analysis); (ii) expanding the logarithm to first order in s^+, J, and the Poisson bracket $\{H, T(\mathbf{x}_\parallel)\}$ (higher order terms of this expansion are small as $1/N$); (iii) performing the $x_\perp$ integration of the resulting action. The last step relies on the fact that within the energy band, where averaging takes place, the classical dynamics is independent of the energy. As a result we obtain the σ model:

$$\langle \mathcal{Z}(J)\rangle_{\epsilon_0} = \int \mathcal{D}T(\mathbf{x}_\parallel)\exp(-\bar{S})\,, \quad (13)$$

$$\bar{S} = \frac{\pi}{ih^d}\int d\mathbf{x}_\parallel\,\mathrm{STr}\left[\left(\frac{s^+}{2} + Jk\right)\Lambda\mathcal{Q} + i\mathcal{Q}T^{-1}\mathcal{L}T\right], \quad (14)$$

where

$$\mathcal{Q} = -\frac{i}{\pi N}\int dx_\perp\, T^{-1}(\mathbf{x}_\parallel)Q_0(x_\perp)T(\mathbf{x}_\parallel) = T^{-1}\Lambda T\,, \quad (15)$$

and $\mathcal{L}$ is the dimensionless infinitesimal time evolution operator defined by the Poisson bracket

$$\mathcal{L}\,\cdot = \hbar\{\cdot, H\}\,. \quad (16)$$

For a stochastic Hamiltonian, an action equivalent to Eq. (14) has been proposed recently using an argument which relies on disorder averaging in the limit of vanishing disorder [10].

To interpret the functional integral in Eq. (13) we must identify the low lying modes of the action. In the case of impurities, the low lying degrees of freedom correspond to the eigenvalues of the diffusion operator that form a discrete spectrum. In the present case it is tempting to associate the low energy degrees of freedom with eigenmodes of the unitary (reversible) evolution operator $e^{-\mathcal{L}t}$. However, this identification is incorrect.

As with any functional integral there is a need to define an appropriate regularization. For example, the functional integral in Eq. (13) may be understood as the limit $a \to 0$ of a product of definite integrations over a discretized space, where a denotes the discretization cell size. This admits to *smooth* and square integrable functions $T(\mathbf{x}_\parallel)$. More generally, a regularization can be performed by truncating an arbitrary complete basis.

In seeking such a basis, the eigenfunctions of the classical evolution operator $\mathcal{L}$ seem to be the natural choice. However, the intricate nature of chaotic classical evolution causes these eigenfunctions to lie generally outside the Hilbert space. Indeed, chaotic dynamics of probability densities involves contraction along stable manifolds, together with stretching along unstable ones. Thus, an initially nonuniform distribution eventually turns into a function singular along the stable manifold, which in turn covers the whole energy shell densely. Therefore the eigenfunctions of $\mathcal{L}$, which require the infinite time limit, are not square integrable and their contribution to the functional integral cannot be recovered by the discretization procedure.

In the present case, a convenient basis is that associated with the classical evolution operator subject to a small noise. The primary effect of the noise is to stop the contraction along the stable manifold, and thereby render the eigenfunctions well behaved. The truncated matrix $e^{-\mathcal{L}t}$, which is nearly diagonal in this basis, is no longer unitary. Apart from one eigenvalue, the modulus of the others, in the limit where the size of this matrix tends to infinity, is smaller than unity. These eigenvalues constitute the PF spectrum and reflect intrinsic irreversible properties of the *purely classical dynamics*. They coincide with those of the evolution operator with noise in the limit where the strength of the noise tends to zero [11]. Other approaches which recover this spectrum involve the use of symbolic dynamics [12], course graining of the flow dynamics in phase space [13], and methods of analytic continuation [14]. We remark that the physical spectrum of the classical operator $\mathcal{L}$ appears when it propagates smooth probability densities. Thus the matrices $T(\mathbf{x}_\parallel)$ on which $\mathcal{L}$ acts are understood to be smooth. In this respect, one can view quantum mechanics as the natural framework for calculating irreversible properties of classically chaotic systems.

Let $\{\gamma_n\}$ be the set of eigenvalues of the generator, $\mathcal{L}$, of the PF operator. In ergodic systems the leading eigenvalue $\gamma_0 = 0$ is nondegenerate, and manifests the conservation of probability density. Thus any initial density eventually relaxes to the state associated with γ_0. If, in addition, this relaxation is exponential in time, then the slowest decay rate determines the gap in the PF spectrum. Thus, for the first nonzero eigenvalue γ_1, we have $\gamma_1' \equiv \Re(\gamma_1) > 0$. This gap sets the ergodic time scale, $\tau_c = 1/\gamma_1'$, over which the classical dynamics relaxes to equilibrium. In the case of disordered metallic grains it coincides with the Thouless time, while in ballistic systems or billiards it is of the order of the time of flight across the system.

In the limit $s \ll \gamma_1'$ the dominant contribution to Eq. (13) comes from the ergodic classical distribution, the zero mode $\mathcal{L} T_0 = 0$. With this contribution alone the functional integral (13) becomes definite,

$$\langle Z(J)\rangle_{\epsilon_0} = \int d\mathcal{Q}_0 \exp\left(i\frac{\pi}{2}\,\mathrm{STr}[(s^+ + 2Jk)\Lambda \mathcal{Q}_0]\right), \tag{17}$$

where $\mathcal{Q}_0 = T_0^{-1}\Lambda T_0$. Equation (17) also follows from RMT and is known to lead to Wigner-Dyson statistics [7,15].

This result can be generalized to any n-point correlator as well as to T-invariant systems. Therefore the quantum statistics of chaotic systems with exponential classical relaxation are described by RMT at energies smaller than γ_1'.

The RMT description is expected to hold even for certain chaotic systems where the PF spectrum is gapless [3] such as the stadium or the Sinai billiards where classical correlation functions decay algebraically in time [16]. The resolvent $1/(z - \mathcal{L})$ in this case may have cuts which reach the $\Im z$ axis. Nevertheless, the RMT description seems to hold whenever the spectral weight of the resolvent inside the strip $0 \le \Re z \le 1$ (which, however, excludes the pole at the origin) is much smaller than unity.

The strength of the field theoretic approach is that it encompasses a range of energy scales which goes well beyond RMT. In particular, procedures similar to those used in Refs. [17,18] allow us to evaluate $R(s)$. For the unitary ensemble

$$R(s) = -\frac{1}{4\pi^2}\frac{\partial^2}{\partial s^2}\ln\mathcal{D}(s) + \frac{\cos(2\pi s)}{2\pi^2}\,\mathcal{D}(s), \tag{18}$$

where $\mathcal{D}(s) = \prod_\mu A_\mu (s^2 + \gamma_\mu^2)^{-1}$ is the spectral determinant (A_μ being regularization factors [19]) associated with the PF spectrum. This confirms the conjecture made in Ref. [19]. Moreover, the leading order correction to the wave function distribution [20] can also be generalized straightforwardly.

In conclusion, we have established the BGS conjecture for chaotic systems with sufficiently fast decay of classical correlation functions and no discrete symmetries. Moreover, the field theoretic approach allows us to study statistical characteristics on a much wider energy scale than that in which RMT applies. These statistics are determined by the analytic properties of the classical resolvent operator $1/(z - \mathcal{L})$. This theory, in principle, offers a systematic controlled way of investigating quantum corrections to the leading semiclassical description.

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