Spectral Statistics: From Disordered to Chaotic Systems

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The relation between disordered and chaotic systems is investigated. It is obtained by identifying the diffusion operator of the disordered systems with the Perron-Frobenius operator in the general case. This association enables us to extend results obtained in the diffusive regime to general chaotic systems. In particular, the two-point level density correlator and the structure factor for general chaotic systems are calculated and characterized. The behavior of the structure factor around the Heisenberg time is quantitatively described in terms of short periodic orbits.

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The statistical description of the quantum spectra of systems, which exhibit chaotic dynamics in their classical limit, has been conducted mainly along two routes. One is to study an *ensemble* of similar systems, such as disordered metallic grains, where electrons experience scattering by a random potential. In this approach, ensemble averaging is a crucial step done at an early stage of the calculation. The results of such a calculation apply to an individual member of the ensemble, provided the time of observation is long enough. The second route is to characterize the properties of *individual* systems by means of the periodic orbit theory [1]. This is possible for a system with chaotic dynamics governed by a Hamiltonian that is simple enough, so that the parameters of the periodic orbits needed for semiclassical spectral analysis can be calculated explicitly. Averaging in this case is usually done over an energy interval which consists of many energy levels [2]. This approach is very powerful in describing the short time behavior of the system, but is faced with significant problems when applied to times of order or bigger than the Heisenberg time $\tau_H = h/\Delta$ or to energies much smaller than the mean level spacing Δ [3]. Despite the obvious differences between the two approaches it is believed that to a large extent both describe the same physics. In this Letter we are applying results of the first approach in order to extend the periodic orbit theory to times close to τ_H.

The relation between ensembles of metallic grains and ensembles of random matrices (RM) [4] is now well understood. The supersymmetric nonlinear σ model [5] actually provides a microscopic justification for the use of RM theory in order to describe the universal features of these systems. This formalism offers a routine way of calculation of a variety of universal correlation functions for all Dyson pure symmetry RM ensembles and for crossovers between them [6]. In view of the growing interest in applying the supersymmetry approach to the investigation of deviations from universality [7–9], it becomes important to understand the connection between the two approaches described above.

The object that we analyze is the dimensionless two-point level density correlator,

$$R(s) = \Delta^2 \langle \rho(E) \rho(E + s\Delta) \rangle - 1 . \qquad (1)$$

Here $\rho(E)$ is the density of states at energy E, Δ is the mean level spacing, and $\langle \cdots \rangle$ represents ensemble averaging in the case of disordered systems, or averaging over some interval of energy E if an individual chaotic system is considered. The universal form of $R(s)$ is especially simple in the unitary case. It is the sum of smooth and oscillating parts [4]: $R(s) = \delta(s) - [1 - \cos(2\pi s)]/2\pi^2 s^2$. The conventional perturbation theory for disordered metals [10] can provide only the smooth part of $R(s)$ [11]. The $s \gg 1$ asymptotics of $R(s)$ in which the oscillatory term, nonanalytic in $1/s$, is retained was recently evaluated in Ref. [8]. This result (for $s \neq 0$) can be still presented as a sum,

$$R(s) = R_P(s) + R_{\mathrm{osc}}(s) , \qquad (2)$$

of a perturbative term $R_P(s)$ and oscillatory one $R_{\mathrm{osc}}(s)$. We rewrite the expression for $R_P(s)$ (see Ref. [11]) as

$$R_P(s) = -\frac{1}{2\alpha\pi^2} \frac{\partial^2}{\partial s^2} \ln[\mathcal{D}(s)] , \qquad (3)$$

where $\alpha = 2$ for the unitary ensemble and $\alpha = 1$ for T-invariant ensembles. $\mathcal{D}(s)$ is the spectral determinant of a classical operator, namely, the diffusion operator

$$\mathcal{D}(s) = \prod_{\mu} A(\epsilon_\mu) (s^2 + \epsilon_\mu^2)^{-1} . \qquad (4)$$

Here ϵ_μ are eigenvalues (in units of Δ) of the diffusion equation in the grain, and $A(\epsilon_\mu)$ is a regularization factor which equals ϵ_μ^2 for $\epsilon_\mu \neq 0$ and unity otherwise [12].

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Surprisingly the oscillatory term $R_{\rm osc}(s)$, which cannot be obtained by a perturbative calculation, is also governed by the same classical spectral determinant $\mathcal{D}(s)$. For example, in the unitary case it has the form

$$R_{\rm osc}(s) = \frac{\cos(2\pi s)}{2\pi^2}\,\mathcal{D}(s)\,. \tag{5}$$

Since $\mathcal{D}(s)$ is purely classical, it is plausible that for any chaotic system there exists a classical operator whose spectral determinant can be identified with $\mathcal{D}(s)$. In what follows we shall identify this operator for general chaotic systems by a semiclassical analysis of relation (3). For the sake of simplicity we shall consider a two dimensional system which belongs to the unitary ensemble.

The semiclassical analysis begins with Gutzwiller's trace formula [1], which expresses the density of states $\rho(E)$ as a sum over the classical periodic orbits

$$\rho(E) = \frac{1}{\Delta} + \Re\frac{1}{\pi\hbar}\sum_p T_p \sum_r \frac{e^{(i/\hbar)S_p(E)r - i\nu_p r}}{|\det(M_p^r - I)|^{1/2}}\,, \tag{6}$$

where p labels a primitive orbit that is characterized by a period T_p, action $S_p(E)$, and Maslov phase ν_p; r stands for the number of the repetitions of this orbit. M_p is the monodromy matrix associated with the linearized dynamics on the Poincaré section perpendicular to the orbit. From here on, energy and time will be measured in units of Δ ($\epsilon = E/\Delta$) and $\hbar/\Delta$, respectively. One can substitute (6) into (1) and represent $R(s)$ in the form of a double sum over the periodic orbits. $R_P(s)$ is given by the diagonal part of this sum. Expanding $S_p(\epsilon + s)$ up to the linear order in s, $S_p(\epsilon + s) \simeq S_p(\epsilon) + T_p s$, we obtain

$$R_P(s) = \Re\frac{1}{2\pi^2}\sum_p T_p^2 \sum_{r=1}^{\infty} \frac{e^{isT_p r}}{|\det(M_p^r - I)|}\,. \tag{7}$$

The traditional way to deal with the above sum is to approximate it by an integral

$$\sum_p \frac{f(T_p)}{|\det(M_p - I)|} \rightarrow \int \frac{dt}{t} f(t) \tag{8}$$

for any sufficiently smooth function $f(t)$. This approximation, known as the Hannay and Ozorio de Almeida (HOA) sum rule [13], holds in the limit $t \rightarrow \infty$ where long periodic orbits which explore the whole energy shell uniformly are considered. In employing it for the calculation of $R_P(s)$, the time t should be restricted to the regime where it is much larger than the shortest periodic orbits but still smaller than the Heisenberg time τ_H. The result associated with it is therefore the universal one $R_P(s) = -1/2\pi^2 s^2$ which holds as long as $s \gg 1$ [2]. Below we present a more careful treatment of the sum (7) that keeps the nonuniversal part of $R_P(s)$.

Let Λ_p be the eigenvalue ($|\Lambda_p| > 1$) of the monodromy matrix M_p. The area preserving property of the latter implies that the second eigenvalue of M_p is $1/\Lambda_p$. Hence,

$$|\det(M_p^r - I)|^{-1} = |\Lambda_p|^{-r}\sum_{k=0}^{\infty}(k + 1)\Lambda_p^{-rk}, \tag{9}$$

and we can rewrite (7) in the form of a triple sum

$$R_P(s) = -\frac{1}{2\pi^2}\frac{\partial^2}{\partial s^2}\Re\sum_{p,k}(k + 1)\sum_{r=1}^{\infty}\frac{1}{r^2}t_{pk}^r\,, \tag{10}$$

where

$$t_{pk} = |\Lambda_p|^{-1}\Lambda_p^{-k}e^{isT_p}. \tag{11}$$

Using the relation (3) we can determine the spectral determinant $\mathcal{D}(s)$ up to a normalization constant:

$$\mathcal{D}(s) = |\mathcal{N}\tilde{Z}(is)|^2. \tag{12}$$

Upon evaluation of the sum over the repetitions in Eq. (10), the expression for $\tilde{Z}(is)$ takes the form

$$1/\tilde{Z}(is) = \prod_p\prod_{k=0}^{\infty}\exp[(k + 1)\phi(t_{pk})], \tag{13}$$

where $\phi(x) = \int_0^x t^{-1}\ln(1 - t)\,dt$. Notice that the normalization constant $\mathcal{N}$ plays no role in the perturbative part of the two-point correlator. We therefore postpone its determination.

Suppose now that all the periodic orbits are very unstable, namely, $|\Lambda_p| \gg 1$ for all p's. In this case $t_{pk} \rightarrow 0$, $\phi(t_{pk}) \rightarrow -t_{pk}$ and $\tilde{Z}(z)$ reduces to the dynamical zeta function [14],

$$1/Z(z) = \prod_p\prod_{k=0}^{\infty}\left(1 - \frac{e^{zT_p}}{|\Lambda_p|\Lambda_p^k}\right)^{k+1}. \tag{14}$$

This function is the spectral determinant associated with the Perron-Frobenius (PF) operator $\mathcal{L}^t$ (also known as Ruelle-Araki or the transfer operator) [15]. $\mathcal{L}^t$ is the classical evolution operator which propagates phase space density for a time $t > 0$. Its kernel is therefore given by

$$\mathcal{L}^t(\vec{y},\vec{x}) = \delta[\vec{y} - \vec{u}(\vec{x};t)], \tag{15}$$

where $\vec{y}$ and $\vec{x}$ are phase space vectors representing coordinates and momenta, and $\vec{u}(\vec{x};t)$ is the point in phase space to which a particle that starts its motion at $\vec{x}$ arrives after time t. The eigenvalues of the PF operator are of the form $e^{-\gamma_\mu t}$. They are associated with the decaying modes of a disturbance in the density of classical particles exhibiting chaotic dynamics, analogous to the diffusion modes of a disordered system. Yet, the difference is that, unlike in the latter case, here γ_μ can appear also in complex conjugate pairs $\gamma = \gamma' \pm i\gamma''$ where $\gamma' \geq 0$. The leading eigenvalue of the PF operator, $\gamma_0 = 0$, corresponds to the conservation of the number of particles. The dynamical zeta function (14) is the spectral determinant associated with the eigenvalues γ_μ. Up to a normalization constant it is given by the product

$$1/Z(z) = \prod_\mu B_\mu(z - \gamma_\mu)\,, \tag{16}$$

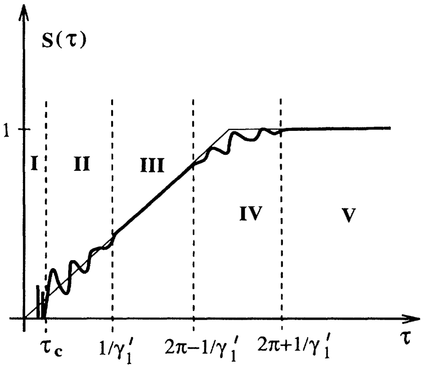

FIG. 1. A schematic drawing of the structure factor of a chaotic system belonging to the unitary ensemble. The light line represents the universal RM theory result. For the sake of clarity, the nonuniversal features have been exaggerated.

where B_μ are regularization factors introduced to make the product converge.

Unlike the periodic orbit theory in quantum mechanics which gives only the leading asymptotics in the limit $\hbar \rightarrow 0$, the periodic orbit expansion (14) of (16) is exact. It is however proper to comment that, in its present form, $Z(z)$ cannot be used to determine the eigenvalues γ_μ. For this purpose a resummed formula is required. It can be obtained by expanding the infinite product over the periodic orbits and ordering the various terms in a way that leads to maximal cancellation among them. This so-called cycle expansion [16] exploits the property that the dynamics of chaotic systems in phase space is coded by a skeleton of few periodic orbits. In particular, the long periodic orbits may be approximated by linear combinations of few short ones.

From (16) and (3) it follows that

$$R_P(s) = \Re \frac{1}{2\pi^2} \sum_\mu \frac{1}{(-is + \gamma_\mu)^2}, \tag{17}$$

in complete analogy with the result of Ref. [11] for diffusive systems. The universal part of $R_P(s)$, which was obtained using the HOA sum rule, thus corresponds to the first term in the sum (17) ($\gamma_0 = 0$). The rest of the sum is apparently system specific.

We turn now to the determination of the normalization constant $\mathcal{N}$ introduced in (12). We shall assume that the leading eigenvalue γ_0 is of unit multiplicity (this is the case when the system is ergodic). Comparison of Eqs. (12) and (4) gives the normalization factor

$$\mathcal{N}^{-1} = \lim_{z \to 0} z Z(z). \tag{18}$$

It is customary to express the semiclassical density of states as the logarithmic derivative of the Selberg zeta function. The latter is defined as the spectral determinant associated with the semiclassical energy spectrum of the system under consideration:

$$\zeta_s(\epsilon) = \prod_j b_j(\epsilon - \epsilon_j) = \prod_p \prod_{k=0}^{\infty} \left(1 - \frac{e^{iS_p(\epsilon) - i\nu_p}}{|\Lambda_p|^{1/2}\Lambda_p^k}\right), \tag{19}$$

where b_j are regularization factors, and ϵ_j are the semiclassical energy levels of the system. The second equality above holds for two-dimensional systems. One can show that the spectral determinant $\tilde{Z}(is)$ satisfies the relation

$$\tilde{Z}(is) = \exp\{\langle \ln[\zeta_s(\epsilon + s)] \ln[\zeta_s^*(\epsilon)] \rangle_d\}. \tag{20}$$

where $\langle \cdots \rangle_d$ represents an averaging which retains only the diagonal elements in the double sum. Since $\Delta\rho(\epsilon) = 1 - (\partial/\pi\partial\epsilon)\Im \ln \zeta_s(\epsilon + i0)$, the two-point correlator can be written as

$$R(s) = -\frac{\partial^2}{\pi^2 \partial s^2} \langle \Im \ln \zeta_s(\epsilon + s) \Im \ln \zeta_s(\epsilon) \rangle. \tag{21}$$

The diagonal approximation gives the perturbative term

$$R_P(s) = -\frac{\partial^2}{\pi^2 \partial s^2} \langle \Im \ln \zeta_s(\epsilon + s) \Im \ln \zeta_s(\epsilon) \rangle_d. \tag{22}$$

The difference between Eqs. (21) and (22) can be also expressed through the diagonal average. Using Eqs.(20) and (12) it is easy to see that $R_{\rm osc}(s)$ is given by Eq. (5) with

$$\mathcal{D}(s) = \mathcal{N}^2 \exp\{2\Re \langle \ln \zeta_s(\epsilon + s) \ln \zeta_s^*(\epsilon) \rangle_d\}. \tag{23}$$

It is convenient to present the result in terms of the Fourier transform of the two-point level density correlator, $S(\tau) = \int ds\, e^{is\tau} R(s)$, known as the structure or the form factor. RM theory predicts that for the unitary ensemble $S(\tau) = \min(|\tau|/2\pi, 1)$ (see the light line in Fig. 1). In the general case

$$S(\tau) = S_P(\tau) + \frac{1}{2}[S_{\rm osc}(\tau + 2\pi) + S_{\rm osc}(\tau - 2\pi)], \tag{24}$$

where S_P and $S_{\rm osc}$ are respectively associated with the perturbative (3) and nonperturbative (5) parts of the two-point correlator. Assuming that the multiplicity of all the eigenvalues γ_μ is 1,

$$S_P(\tau) = \frac{|\tau|}{2\pi} \sum_\mu e^{-\gamma_\mu |\tau|}. \tag{25}$$

Again, the universal part of $S_P(\tau)$ associated with the HOA sum rule comes from the leading eigenvalue $\gamma_0 = 0$. The higher eigenvalues will contribute corrections which are in general oscillatory and decrease exponentially. For instance, the complex pair $\gamma_1' \pm i\gamma_1''$ will contribute the term $|\tau| e^{-\gamma_1' |\tau|} \cos(\gamma_1'' \tau)/\pi$. The oscillatory part of the structure factor can be written as

$$S_{\rm osc}(\tau) = -\frac{|\tau|}{2\pi} - \sum_{\mu \neq 0} \frac{\mathcal{D}_\mu(i\gamma_\mu)}{2\pi\gamma_\mu} e^{-\gamma_\mu |\tau|}, \tag{26}$$

where $\mathcal{D}_\mu(s)$ is given by

$$\mathcal{D}_\mu(s) = \left(1 + \frac{s^2}{\gamma_\mu^2}\right) s^2 \mathcal{D}(s). \tag{27}$$

For example, in the case of a quasi-one-dimensional diffusive system, where the eigenvalues are of the form $\gamma_n = Dn^2$ one can show that $\mathcal{D}_n(iDn^2) = -4n(-1)^n/\sinh(\pi n)$, while for equally spaced eigenvalues $\gamma_n = vn$ it is $\mathcal{D}_n(ivn) = 2\pi n/\sinh(\pi n)$. In general it is expected that the contribution will come only from the lowest eigenvalues of the PF operator.

In what follows it will be assumed that the nonuniversal behavior is dominated by one eigenvalue (or possibly a conjugate pair) γ_1, i.e., $\gamma'_\mu \gg \gamma'_1$ for all $\mu > 1$.

In characterizing $S(\tau)$, five domains of the parameter τ, drawn schematically in Fig. 1, are identified: (I) $\tau \sim \tau_c$, where τ_c is of order of the period of the shortest periodic orbit. Here $S(\tau)$ is composed of several δ-function peaks located at the periods of the short orbits and weighted according to their instability. (II) $\tau_c < \tau < 1/\gamma'_1$, deviations from universality associated with (25) may be noticeable also in this interval. Their period of oscillation $1/\gamma''_1$ is of the order of τ_c. (III) $1/\gamma'_1 < \tau < 2\pi - 1/\gamma'_1$, the universal perturbative regime where $S(\tau) = \tau/2\pi$. This is the domain where the HOA sum rule holds. (IV) $2\pi - 1/\gamma'_1 < \tau < 2\pi + 1/\gamma'_1$, the vicinity of the Heisenberg time $\tau = 2\pi$. The nonuniversal features here are in the form of exponentially decreasing oscillations very similar to those existing in (II). Yet their amplitude and phase may be different. In general, the RM singularity at the Heisenberg time (the light line in Fig. 1) will be smeared out by them. (V) $\tau > 2\pi + 1/\gamma'_1$, here again the universal result $S(\tau) = 1$ holds.

These results can be generalized straightforwardly to orthogonal and symplectic chaotic systems: instead of Eq. (5) one should use Eq. (5) of Ref. [8] with $P(s) = s^2\mathcal{D}(s)$.

The behavior of the structure factor in the vicinity of the Heisenberg time is a manifestation of a striking property of the periodic orbit sum (6), namely, that the tail of the Gutzwiller's series (the long periodic orbit) encodes its head (short periodic orbit). As a result, $S(\tau)$ in the vicinity of the Heisenberg time is determined by the same short periodic orbits as at small τ. The argument [2] is that the long periodic orbits determine the position of the energy levels. Therefore through the long range correlation of these levels they encode the information about the short periodic orbits. In fact, the Berry-Keating resummation method [17] of the periodic orbit sum associated with the quantum spectral determinant of chaotic systems is based on the bootstrapping of long periodic orbits with periods near the Heisenberg time $\tau \sim 2\pi$ to the short ones near τ_c. The behavior of $S(\tau)$ near the Heisenberg time reflect this sort of symmetry in the sense that it is determined by the short time dynamics of the classical system.

To summarize, we identified the diffusion operator in disordered grains with the Perron-Frobenius operator in the general case. This relates the spectral determinant associated with the diffusion equation in the grain to the dynamical zeta function which can be expressed in terms of the classical periodic orbits. We used these relations to extend the theory of the structure factor of disordered grains to general chaotic systems. It would be interesting and important to derive these relations for generic chaotic systems. In this respect the recently proposed σ-model-like approach for ballistic systems [18] looks promising.

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