ADVANCED ASTROPHYSICS (77938) - PROBLEM SET NO. 2

Due Date: May 10, 2012

1. Mixed star

We examine a star at the limits of $L \ll L_{Edd}$ and $P_{rad} \ll P_{gas}$. In objects such as this, mixing of the interior will occur changing the chemical composition of the star. How would this effect the luminosity of the star over its lifetime?

a. Find the Hydrogen fraction X in the sun today. Assume that the Sun was formed from a primordial composition of Hydrogen and Helium, $X = 0.75$.

   Hint: Assess the change in X and Y over the lifetime of the sun 4.5Gyr, via the nuclear reaction $4\text{H} \to \text{He}$, given the solar luminosity $L_\odot = 3.8 \times 10^{33}$erg/sec.

b. Assume that the sun can be described by a Polytropic model of $n = 3$ (aka the Standard or Eddington Model). For such a model we have shown that $\beta = P_g/P = \text{const.}$ within the star. Use the expressions derived in class for mass and luminosity for this model to relate the luminosity of a star to its composition.

   Hint: Remember that $\beta = \beta(M)$.

c. By how much has the luminosity of the Sun changed over the years as a result?

2. Massive Main Sequence Star

Let us approximately describe a main-sequence massive star as an approximate $n = 3$ polytrope. Note that such stars have a significant radiation pressure. Assume also that the opacity is the Thomson opacity and that the energy generation per unit mass (by the dominant CNO cycle) is given approximately by:

$$\epsilon_{CNO} \approx 2.1 \times 10^{15} XZ \left(\frac{T}{10^8 K}\right)^8 \left(\frac{\rho}{gr\ cm^{-3}}\right) erg\ gr^{-1}\ s^{-1}. \tag{1}$$

Using the polytropic relations, first find $L_{\text{nuclear}}(R)$. However, given that there is a second relation $L(M, R)$ for the radiative transfer in the star, find $L(M)$ and $R(M)$. You can leave the answer with dimensionless integrals over the $\phi^p \xi^q$, or use the mathematica notebook in the website.

3. Iron White Dwarf

A non-relativistic white dwarf composed entirely of carbon ${}^{12}_{6}C$ (6 protons and $12 - 6$ neutrons) has a solution $P(r), \rho(r)\, \&\, m(r)$ and a total mass of M.

a. What would be the solution $\tilde{P}(r), \tilde{\rho}(r)\, \&\, \tilde{m}(r)$ of a white dwarf of the same total mass composed entirely of ${}^{56}_{26}Fe$? Express you answer with $P(r), \rho(r)\, \&\, m(r)$

b. What is the Chandrasekhar mass for a ${}^{12}_{6}C$ white dwarf? And for a ${}^{56}_{26}Fe$ white dwarf?

c. What would be the Chandrasekhar mass of a star composed entirely of degenerate neutrinos. Assume the neutrino mass is 10^{-3}eV

4. Uniformly Rotating Star

Consider a chemically homogeneous star that is in hydrostatic and thermal balance. Assume it is rigidly rotating about some axis with an angular velocity Ω. The equation of state is that of an ideal gas and the energy transport is through radiative diffusion. Show that the above considerations require the nuclear energy production rate to satisfy

$$\epsilon \propto \left(1 - \frac{\Omega^2}{2\pi G\rho}\right)$$

a. The condition of hydrostatic equilibrium in the non-rotating case can be written

$$\vec{\nabla} P = -\rho\vec{\nabla}\phi_g$$

Where ϕ_g is the gravitational potential. Show that this form can be extended to the rotating case by adding a 'rotational' potential ϕ_r, such that $\phi = \phi_g + \phi_r$. What is the form of ϕ_r?

Hint: The problem can no longer be treated as purely radial.

b. Convince yourself that all the relevant properties, P, ρ, T, $\mathrm{etc.}$ are constant along equipotential surfaces, or in other words, are functions of ϕ alone.

c. As a result, show that the energy flux is proportional to $\vec{\nabla}\phi$. Relate the energy flux to the energy production rate ϵ and find the relation between ϵ and ϕ.

5. Kramer Opacity

a. Show that for a radiation field described by the Planck distribution in which the mean free path for a photon of frequency ν is $l_\nu \propto \rho^{-2}\nu^3 T^{1/2}$, the average opacity (á la Rosseland) follows Kramer's relation

$$\kappa \propto \rho T^{-7/2}$$

Hint: Remember that the mean free path is $l = (\kappa\rho)^{-1}$

b. Using the Virial Theorem, find the relation $L(M)$ (or rather what is the typical exponent $L \propto M^\gamma$), if the opacity is indeed given by $\kappa \propto \rho T^{-7/2}$.